SS 2016 Dr. Milena Pabiniak

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# Differential Geometry

Homework 11

**Mandatory Exercise 1.** (10 points)

(a) Let $X$ and $Y$ be vector fields on $\mathbb{R}^n$. By canonical identification of $T_p\mathbb{R}^n$ with $\mathbb{R}^n$ one can see a vector field as a map $\mathbb{R}^n \to \mathbb{R}^n$. Recall that the derivative of $Y$ in direction of $X$ is defined as

$$D_X Y(p) := \lim_{t\to 0} \frac{Y\big(p + tX(p)\big) - Y\big(p\big)}{t}.$$

Show that this defines the Levi-Civita connection $\nabla^{euc}$ of $\mathbb{R}^n$ with euclidean metric $g_{euc}$.

Now let $M$ be a submanifold of $\mathbb{R}^n$ with induced Riemannian metric $g$.

(b) Let $X$ and $Y$ be vector fields on $M$ and $\tilde{X}$, $\tilde{Y}$ be extensions to $\mathbb{R}^n$. Show that the orthogonal projection of the derivative of $\tilde{Y}$ in direction of $\tilde{X}$ to $T_pM$ represents the Levi-Civita connection of $M$, i.e.

$$\nabla^M_X Y = (D_{\tilde{X}}\tilde{Y})^{\perp T_pM}.$$

(c) Use this result to compute again $\nabla_{\partial_\alpha}\partial_\beta$ from Exercise 1 on Sheet 8.

(d) Show that a curve $c\colon I \to M$ is a geodesic, if and only if its acceleration is orthogonal to $M$.

(e) Use this to find all geodesics on $S^2$ and on the standard embedding of $T^2$ into $\mathbb{R}^3$ ($\phi_2$ from Exercise 1 on Sheet 8).

**Mandatory Exercise 2.** (10 points)
Let $f, g\colon \mathbb{R}^2 \to \mathbb{R}^2$ be isometries given by

$$f(x,y) = (-x, y+1) \text{ and } g(x,y) = (x+1, y).$$

To which manifolds are $\mathbb{R}^2/\langle f\rangle$ and $\mathbb{R}^2/\langle f, g\rangle$ homeomorphic?

**Suggested Exercise 1.** (0 points)
Let $H^2$ be the hyperbolic plane. Show that:

(a) The formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az+b}{cz+d}$$

defines an action of $\mathrm{PSL}_2(\mathbb{R}) := \mathrm{SL}_2(\mathbb{R})/\{\pm \mathrm{Id}\}$ on $H^2$ by orientation preserving isometries.

(b) For any two geodesics $c_1, c_2\colon \mathbb{R} \to H^2$, parametrized by arclength, there exists $g \in \mathrm{PSL}_2(\mathbb{R})$ such that $c_1(s) = g \cdot c_2(s)$ for all $s \in \mathbb{R}$.

(c) Given $z_1, z_2, z_3, z_4 \in H^2$ with $d(z_1, z_2) = d(z_3, z_4)$, there exists $g \in \mathrm{PSL}_2(\mathbb{R})$ such that $g \cdot z_1 = z_3$ and $g \cdot z_2 = z_4$.

(d) An orientation preserving isometry of $H^2$ with two fixed points must be the identity. Conclude that all orientation preserving isometries are of the form $f(z) = g\cdot z$ for some $g \in \mathrm{PSL}_2(\mathbb{R})$.

b.w.

**Suggested Exercise 2.** (0 points)
Prove the second Bianchi identity:

For any vector fields $X, Y, Z, T$ it holds that

$$(\nabla_X R)(Y,Z,T) + (\nabla_Y R)(X,Z,T) + (\nabla_Z R)(X,Y,T) = 0.$$

Here $(\nabla_X R)$ denotes the covariant derivative of the Riemann $(1,3)$ tensor field, in the direction of $X$, that is,

$$(\nabla_X R)(Y,Z,T) = \nabla_X(R(Y,Z)T) - R(\nabla_X Y, Z)T) - R(Y, \nabla_X Z)T - R(Y,Z)\nabla_X T.$$

Hint: Viewing $\nabla_X \_$, $(\nabla_X R)(Y, Z, \_)$, $R(Y,Z)$, etc. as maps from $\mathfrak{X}(M)$ to $\mathfrak{X}(M)$, and using the notation $[f,g] = f \circ g - g \circ f$, prove that

$$(\nabla_X R)(Y, Z, \_) = [\nabla_X, R(Y,Z)] - R(\nabla_X Y, Z) - R(Y, \nabla_X Z).$$

Then use $R(Y,Z) = [\nabla_Y, \nabla_Z] - \nabla_{[Y,Z]}$. Permute $X, Y, Z$ cyclically and add all these three equations. Then the Jacobi identity on Lie brackets, and the fact that the connection is torsion free, imply the second Bianchi identity.

**Suggested Exercise 3.** (0 points)
Let $M^n$ be a connected, $n$-dimensional Riemannian manifold, with $n \geq 3$. Assume that $M$ is isotropic, i.e. there exists a smooth function $K$ on $M$ such that for any $p \in M$, and any 2-dimensional subspace $H$ of $T_pM$, the sectional curvature $K(H) = K(p)$. Then $M$ is of constant sectional curvature, i.e. function $K$ is constant.

Hints:

- Recall that we had shown in class that if $M$ is isotropic then

  $$R(X,Y,Z,W)_p = -K(p)R'(X,Y,Z,W)_p$$

  where

  $$R'(X,Y,Z,W) = \langle X,Z\rangle \cdot \langle Y,W\rangle - \langle X,W\rangle \cdot \langle Y,Z\rangle.$$

  Show that for any $U \in \mathfrak{X}(M)$ we have that $\nabla_U R = -U(K) \cdot R'$.

- To prove that $K$ is constant it is enough to show that for any $p \in M$ and any $X \in \mathfrak{X}(M)$ one has $X_p(K) = 0$. To show this, take any $X \in \mathfrak{X}(M)$ and fix any $p \in M$. As $\dim M \geq 3$, one can find vector fields $Y, Z \in \mathfrak{X}(M)$ such that $\langle X,Y\rangle = \langle Y,Z\rangle = \langle X,Z\rangle = 0$ and $\langle Z,Z\rangle = 1$ around $p$.

- The Second Bianchi Identity, proved in previous exercise, gives that for any $W \in \mathfrak{X}(M)$

  $$0 = \langle (\nabla_Z R)(X,Y,Z) + (\nabla_X R)(Z,Y,Z) + (\nabla_Y R)(Z,X,Z), W\rangle.$$

  Use the first hint to deduce from the above that $X(K) \cdot Y + Y(K) \cdot X = 0$ around $p$. Then deduce that $X_p(K) = 0$.

Hand in: Monday 4th July
in the exercise session
in Seminar room 2, MI